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New Topological Invariants for Non-Abelian Antisymmetric Tensor Fields from extended BRS Algebra*

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Abstract

Extended non-linear BRS and Gauge transformations containing Lie algebra cocycles, and acting on non-abelian antisymmetric tensor fields are constructed in the context of free differential algebras. New topological invariants are given in this framework.

BRS transformations are known to yield the relevant symmetries of the quantum action of gauge theories. As such, the BRS symmetries have turned out to play a major role in the understanding of the occurrence of anomalies and of their geometrical interpretation, see for example ref. [1], and references therein. We will present in this note an extension of the usual BRS and gauge transformations, for non-abelian antisymmetric tensor fields, and show how it can be used to construct new topological invariants for these p-form gauge fields. A more detailed description of most of the material presented here, together with an extended list of references may be found in ref. [2]. However, some ideas and results, that may be useful in the context of topological quantum field theories[3] are new. Our starting point is the construction of a BRS algebra for p-form gauge fields in the frame

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work of free differential algebra (FDA)[4,5]. We consider a graded commutative FDA $\tilde{\mathcal{M}}$[4,6], connected in degree zero, i.e., $\tilde{\mathcal{M}} = \bigoplus_{p\geq o} \tilde{\mathcal{M}}_p = \tilde{\mathcal{M}}_0 \bigoplus \tilde{\mathcal{M}}_+, \tilde{\mathcal{M}}_0 = \mathbf{K}$ being the ground field ($\mathbf{R}$ or $\mathbf{C}$). We denote the differential by $\tilde{d}$. We now require that $\tilde{\mathcal{M}}$ and $\tilde{d}$ can be decomposed as $\tilde{d} = d + s$ with $d^2 = s^2 = sd + ds = 0$ and $\tilde{\mathcal{M}} = \bigoplus_{p,q} \tilde{\mathcal{M}}_{p,q}$, where $p \geq 0, q \geq 0$ p being the form degree with respect to d, and q the form degree with respect to s. In addition, we also require that the sub-algebra $\mathcal{M}$ of $\tilde{\mathcal{M}}$ corresponding to the action of $s, \mathcal{M} = \bigoplus_{q\geq 0} \tilde{\mathcal{M}}_{o,q}$ is minimal, i.e., $s\mathcal{M}_{0,+} \subset \mathcal{M}_{0,+} \cdot \mathcal{M}_{0,+}, \mathcal{M}_{0,+} = \bigoplus_{q\geq 1} \tilde{\mathcal{M}}_{0,q}$. A minimal FDA is obtained[5] by extending the Maurer–Cartan forms in the dual $\mathcal{G}^*$ of a Lie (super)–algebra $\mathcal{G}$, identifying $\wedge\mathcal{G}^*$ with $\mathcal{M}_{0,1}$, and by adding new generators in degrees higher than one. Choosing a basis $\{\chi_1^\alpha\}$ of $\mathcal{G}^*$ and representations $\mathcal{G}(p)$ of $\mathcal{G}(\mathcal{G}_{(1)} = \mathcal{G})$, we introduce at each level $p \geq 2$ sets of new generators χ_p^i. Now the subspace ζ of $\mathcal{M}_{0,+}^*$ (dual of $\mathcal{M}_{0,+}$) defined by $\zeta = \left\{w\epsilon\mathcal{M}_{0,+}^*; w(a_1 \cdot a_2) = 0, \forall(a_1, a_2)\epsilon\mathcal{M}_{0,+}\right\}$, a canonical Lie–(super) algebraic structure, with (graded) basis $E_i^{(p)}, p \geq 1$, and we have $\tilde{d}E_i^{(p)} = (-1)^{p+1}E_i^{(p)}\tilde{d}$. We define elements of $\mathcal{M}_{0,+} \otimes \zeta$ by $\chi = \sum_{i,p} \chi_p^i E_i^{(p)}$. Note that χ anticommute with $\tilde{d}$. The more general action of s on χ, $\mathcal{M}$ being minimal, reads:

$$s\chi + \sum_{p\geq 2} \frac{1}{p!} C^{(p)}(\chi, \cdots, \chi) = 0 \tag{1}$$

where we have used multilinear maps $C^{(p)} : \wedge^p\zeta \rightarrow \zeta, p \geq 2$, which for consistency, ($s^2 = 0$), verify p–cocycle conditions with respect to $\mathcal{G}$ and $\mathcal{G}(p), p \geq 2$:

$$\sum_{p,q\geq 2} \frac{1}{(p-1)!q!} C^{(p)}\left(\chi, \cdots, \chi, C^{(q)}(\chi, \cdots, \chi)\right) = 0 \tag{2}$$

Eq. (1) is a generalization of the Maurer–Cartan equation, with $C^{(p)}$ extended to $\mathcal{M}_{0,+} \otimes \zeta$. Now we will consider the case of a FDA $\tilde{\mathcal{M}}$ with generators $\tilde{\mathcal{A}}_{p,q}^i$ and $\tilde{\mathcal{F}}_{p',q'}^i$ in degree k ($p' + q' - 1 = p + q = k$) such that $\tilde{\mathcal{A}}_{0,q}^i \equiv \chi_q^i$. Then defining $\tilde{\mathcal{A}} = \sum_{i,p,q} \tilde{\mathcal{A}}_{p,q}^i E_i^{(p+q)}$ and $\tilde{\mathcal{F}} = \sum_{i,p,q} \tilde{\mathcal{F}}_{p,q}^i E_i^{(p+q-1)}$, the action of $\tilde{d}$ in $\tilde{\mathcal{M}}$ is given by:

$$\tilde{d}\tilde{\mathcal{A}} + C_{\tilde{\mathcal{A}}}^{(0)} = \tilde{\mathcal{F}} \text{ and } \tilde{d}\tilde{\mathcal{F}} + C_{\tilde{\mathcal{A}}}^{(1)}(\tilde{\mathcal{F}}) = 0 \tag{3}$$

where we have introduced the following compact notations:

$$C_{\tilde{\mathcal{A}}}^{(q)}(\nu_1, \cdots, \nu_q) = \sum_{p\geq q} \frac{1}{(p-q)!} C^{(p)}(\tilde{\mathcal{A}}, \cdots, \tilde{\mathcal{A}}, \nu_1, \cdots, \nu_q)$$

Then, eq. (3) may be recasted in a very familiar form by using an operator formulation that may be viewed as a generalized adjoint–action on $\tilde{\mathcal{M}} \otimes \zeta$; in fact, we define $\hat{\tilde{\mathcal{A}}}$ and $\hat{\tilde{\mathcal{F}}}$ to be linear operators on $\tilde{\mathcal{M}} \otimes \zeta$ considered as a vector space as:

$$\hat{\tilde{\mathcal{A}}}\,(\nu) = C^{(1)}_{\tilde{\mathcal{A}}}(\nu), \quad \hat{\tilde{\mathcal{F}}}\,(\nu) = C^{(2)}_{\tilde{\mathcal{A}}}\left(\tilde{\mathcal{F}}, \nu\right)$$

$$\tilde{d}\,\hat{\tilde{\mathcal{A}}} + \hat{\tilde{\mathcal{A}}} \cdot \hat{\tilde{\mathcal{A}}} = \hat{\tilde{\mathcal{F}}} \quad \text{and} \quad \tilde{d}\,\hat{\tilde{\mathcal{F}}} + \hat{\tilde{\mathcal{A}}} \cdot \hat{\tilde{\mathcal{F}}} - \hat{\tilde{\mathcal{F}}} \cdot \hat{\tilde{\mathcal{A}}} = 0 \tag{4}$$

where the products in eq. (4) are operator products. A decomposition of $\tilde{d}, \tilde{\mathcal{A}}$ and $\tilde{\mathcal{F}}$ in d and s degrees leads to the d and s action on the $\tilde{\mathcal{M}}$ generators[2]. Note that because of the minimality of $\mathcal{M}$, (the FDA associated to s) we have $\tilde{\mathcal{F}}^i_{0,q} \equiv 0$.

We will obtain a BRS algebra from this bigraded FDA. We wish, in fact, to consider s as the generator of BRS transformations. Hence, the degrees of form with respect to s will become ghost degrees. Therefore, we identify the physical field as the zero s–degree part of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{F}}$, i.e., $\mathcal{A} = \sum_{i,p\geq 1} \tilde{\mathcal{A}}^i_{p,0} E^{(p)}_i$ and $\mathcal{F} = \sum_{i,p\geq 2} \tilde{\mathcal{F}}^i_{p,0} E^{(p-1)}_i$, the generalized gauge field and curvature respectively. We define $\psi = \sum_{i,p\geq 0} \tilde{\mathcal{A}}^i_{p,1} E^{(p+1)}_i$ and $\theta = \sum_{i,p\geq 1} \mathcal{F}^i_{p,1} E^{(p)}_i$, the ghost degree one part of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{F}}$, we have:

$$s\mathcal{A} + d\psi + C^{(1)}_{\mathcal{A}}(\psi) = \theta \tag{5}$$

Hence, the s–transformation of $\mathcal{A}$ contains a linear ghost term θ. That means that the general FDA $\tilde{\mathcal{M}}$ cannot be taken directly as the BRS algebra associated to $\mathcal{A}$ and $\mathcal{F}$. The reason is that the linear term θ in eq. (5) would enable us to gauge the fields $\mathcal{A}$ and $\mathcal{F}$ to zero, therefore, leading to a pure gauge theory. In fact, the space of s–transformations in $\tilde{\mathcal{M}}$ is so large that only a pure gauge theory would be s–invariant. The idea for obtaining a non–trivial theory is indeed to reduce the number of degrees of freedom of θ and ψ by imposing a set of constraints invariant by the actions of d and s. The procedure is first to construct bigraded ideals of $\tilde{\mathcal{M}}$ and then to quotient $\tilde{\mathcal{M}}$ by such bigraded ideals. The resulting algebra is again a bigraded differential algebra in which the restriction of d and s are uniquely defined and verify $s^2 = d^2 = sd + ds = 0$. Such ideals can be obtained, for example, by considering any element K of the s–cohomology in $\tilde{\mathcal{M}}$, i.e., $sK = 0$, of homogeneous s and d degrees, constructed from $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{F}}$. The subalgebra $\mathcal{T}(K)$ generated by elements of the form $K \cdot P + dK \cdot Q$ where$P$ and Q are any elements of $\tilde{\mathcal{M}}$ is an ideal of $\tilde{\mathcal{M}}$, stable under the action of d and s. The quotiented algebra $\tilde{\mathcal{M}}(K) = \tilde{\mathcal{M}}/\mathcal{T}(K)$ exists and the action of d and s can be defined uniquely on it. $\tilde{\mathcal{M}}(K)$ is also a bigraded differential algebra. The ideal that leads to a non–trivial gauge theory for a usual 1–form gauge field is given by $\tilde{\mathcal{F}}^{i}_{1,q} \equiv 0$. Let us note the quotiented algebra in this case by $\mathcal{D}_1$. The $\tilde{d}$ and d cohomology are trivial in $\mathcal{D}_1$ and it is possible to construct $\tilde{d}$ invariant in $\mathcal{D}_1$ in the following way. As operators acting on a vector space, it is possible to define a (super)–trace for polynomials of $\hat{\tilde{\mathcal{A}}}$ and $\hat{\tilde{\mathcal{F}}}$. Then, we have $\tilde{d}\, tr(\hat{\tilde{\mathcal{F}}}^{n}) = 0$. The $\tilde{d}$–cohomology being trivial we obtain:

$$I^{(n)} = tr(\hat{\tilde{\mathcal{F}}}^{n}) = \tilde{d}J^{(n)} \tag{6}$$

where the $J^{(n)}$ may be obtained through standard homotopy formulas transposed to our case. Expanding $I^{(n)}$ and $J^{(n)}$ with respect to their ghost degrees (q) i.e., $I^{(n)} = \sum_{q\geq 0} I^{(n)}_q$ and $J^{(n)} = \sum_{q\geq 0} J^{(n)}_q$, the constraint $I^{(n)}_q = 0$ for $q \geq 1$ is a bigraded ideal J of $\mathcal{D}_1$, stable under the d and s actions. In the BRS algebra $\mathcal{D}_1/J$ we have:

$$dJ^{(n)}_0 = I^{(n)}_0 \ \text{ and } \ sJ^{(n)}_0 + dJ^{(n)}_1 = 0 \tag{7}$$

Hence, $J^{(n)}_0$ is an s–invariant modulo d, containing the p–form gauge fields $\mathcal{A}$ and $\mathcal{F}$, that generalizes the Chern–Simons terms[6]. The BRS algebra associated with topological theories constructed from $I^{(n)}_0$ or $J^{(n)}_0$ may be obtained from $\mathcal{D}_1$ by quotienting with an appropriate ideal.

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